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# LOWER BOUNDS FOR RANKS OF MUMFORD–TATE GROUPS

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ABSTRACT. — Let  $A$  be a complex abelian variety and  $G$  its Mumford–Tate group. Supposing that the simple abelian subvarieties of  $A$  are pairwise non-isogenous, we find a lower bound for the rank of  $G$ , which is a little less than  $\log_2 \dim A$ . If we suppose further that  $\text{End } A$  is commutative, then we show that  $\text{rk } G \geq \log_2 g + 2$ , and this latter bound is sharp. We also obtain the same results for the rank of the  $\ell$ -adic monodromy group of an abelian variety defined over a number field.

RÉSUMÉ (*Minoration des rangs de groupes de Mumford–Tate*). — Soit  $A$  une variété abélienne complexe et  $G$  son groupe de Mumford–Tate. En admettant que les sous variétés abéliennes simples de  $A$  sont deux à deux non-isogènes, on trouve une minoration du rang de  $G$ , un peu moins que  $\log_2 \dim A$ . Si on suppose en plus que  $\text{End } A$  soit commutatif, alors on montre que  $\text{rk } G \geq \log_2 g + 2$ , et cette borne-ci est la meilleure possible. On obtient les mêmes résultats pour le rang du groupe de monodromie  $\ell$ -adique d’une variété abélienne définie sur un corps de nombres.

## 1. Introduction

Let  $A$  be a complex abelian variety of dimension  $g$ , whose simple abelian subvarieties are pairwise non-isogenous. In this paper we will establish a lower bound for the rank of the Mumford–Tate group of  $A$ . The Mumford–Tate group is an algebraic group over  $\mathbb{Q}$  defined via the Hodge theory of  $A$  (see section 2 below for the definition). The same argument will also establish a lower bound for the rank of the  $\ell$ -adic monodromy groups  $G_\ell$ , in the case where  $A$  is defined over a number field. The  $\ell$ -adic monodromy group is the Zariski closure of the image of the Galois representation on the  $\ell$ -adic Tate module of  $A$ . Our main theorems are the following:

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**THEOREM 1.1.** — *Let  $A$  be an abelian variety of dimension  $g$  such that  $\text{End } A$  is commutative. Let  $G$  be the Mumford–Tate group or the  $\ell$ -adic monodromy group of  $A$ . Then  $\text{rk } G \geq \log_2 g + 2$ .*

**THEOREM 1.2.** — *Let  $A$  be an abelian variety of dimension  $g$  whose simple abelian subvarieties are pairwise non-isogenous. Let  $G$  be the Mumford–Tate group or the  $\ell$ -adic monodromy group of  $A$ . If  $n = \text{rk } G$ , then*

$$n + \alpha(n)\sqrt{n \log_e n} \geq \log_2 g + 2$$

*for a function  $\alpha : \mathbb{N}_{\geq 2} \rightarrow \mathbb{R}$  satisfying  $\alpha(n) < 2$  for all  $n$  and  $\alpha(n) \rightarrow 1/\log_e 2 = 1.44\dots$  as  $n \rightarrow \infty$ .*

Theorem 1.1 was proved by Ribet in the case of an abelian variety with complex multiplication [Rib80]. Our proof is a generalisation of his, relying on the fact that the defining representation of the Mumford–Tate group or  $\ell$ -adic monodromy group has minuscule weights.

The condition on simple subvarieties in Theorem 1.2 is necessary: taking products of copies of the same simple abelian variety increases the dimension without changing the rank of the Mumford–Tate group. The condition can be interpreted via  $\text{End } A$  like that in Theorem 1.1: it is equivalent to  $\text{End } A \otimes_{\mathbb{Z}} \mathbb{Q}$  being a product of division algebras. (Throughout this paper,  $\text{End } A$  means the endomorphisms of  $A$  after extension of scalars to an algebraically closed field.)

Let  $G$  be either the Mumford–Tate group or the  $\ell$ -adic monodromy group of  $A$ . It is well known that the rank of  $G$  is at most  $g + 1$ , and that this upper bound is achieved for a generic abelian variety. Indeed, if  $g$  is odd and  $\text{End } A = \mathbb{Z}$ , then  $\text{rk } G$  is always  $g + 1$  [Ser85]. So in this case the bound in Theorem 1.1 is far from sharp.

On the other hand if  $g$  is a power of 2, then there are abelian varieties for which the bound in Theorem 1.1 is achieved (even with  $\text{End } A = \mathbb{Z}$ ). We construct such examples in section 5. The exact bound for a given  $g$  is very sensitive to the prime factors of  $g$ . Equality can happen only when  $g$  is a power of 2 (for the trivial reason that otherwise  $\log_2 g \notin \mathbb{Z}$ ) but even near-equality can only occur when  $g$  has many small prime factors. This was made precise by Dodson in the complex multiplication case [Dod87], and it is possible that something similar could be proved in general.

Theorem 1.2 is not sharp. The function  $\alpha(n)$  is specified exactly in section 4, but it is likely that this could be improved on, perhaps to something which goes to 0 as  $n \rightarrow \infty$ . In section 5, we construct a family of examples showing that Theorem 1.2 cannot be improved to  $n + k \geq \log_2 g$  for any constant  $k$ .

We can deduce a lower bound for the growth of the degrees of the division fields  $K(A[\ell^n])$  (for  $\ell$  a fixed prime number) as a straightforward consequence of Theorem 1.1.

**COROLLARY 1.3.** — *Let  $A$  be an abelian variety of dimension  $g$  over a number field  $K$ , and  $\ell$  a prime number. If  $\text{End } A$  is commutative, then there is a constant  $C(A, K, \ell)$  such that*

$$[K(A[\ell^n]) : K] \geq C(A, K, \ell) \ell^{n(\log_2 g + 2)}.$$

Theorem 1.2 implies a similar bound for the degree of  $K(A[\ell^n])$  whenever  $A$  is an abelian variety whose simple abelian subvarieties are pairwise non-isogenous. One would like to extend these results to lower bounds on the degrees of  $K(A[N])$  for any  $N$ , but this cannot be done without knowing how  $C(A, K, \ell)$  varies with  $\ell$ . The primary obstacle here is the index of the image of  $\text{Gal}(\bar{K}/K)$  in  $G_\ell(\mathbb{Z}_\ell)$ , which is conjectured to be bounded by a constant  $C_1(A, K)$  independent of  $\ell$ .

In section 2 we recall the definitions of Mumford–Tate group,  $\ell$ -adic monodromy group and Mumford–Tate triple, an axiomatisation of the properties of the groups and representations we will consider. In section 3 we bound the number of distinct characters of a maximal torus which can appear in such a representation. This is enough to imply Theorem 1.1. In section 4 we bound the multiplicity of absolutely irreducible components of this representation. This is combined with the bound from section 3 to obtain Theorem 1.2. Finally in section 5 we give some examples to show that Theorem 1.1 is sharp and to place a limit on the possible improvement of Theorem 1.2.

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## 2. Mumford–Tate triples: Definitions

We recall the definition of a weak Mumford–Tate triple, which abstracts the key properties of a Mumford–Tate group which we will use. We recall also the definitions of the two examples of Mumford–Tate triple we will consider, namely the Mumford–Tate group and the  $\ell$ -adic monodromy group of an abelian variety.

The following definition is a slight modification of those used by Serre [Ser79] and Wintenberger [Win86].

**DEFINITION.** — Let  $F$  be a field of characteristic zero and  $E$  an algebraically closed field containing  $F$ .

A *weak Mumford–Tate triple* is a triple  $(G, \rho, \Psi)$  where  $G$  is an algebraic group over  $F$ ,  $\rho$  is a rational representation of  $G$  and  $\Psi$  is a set of cocharacters of  $G \times_F E$  satisfying the following conditions:

- (i)  $G$  is a connected reductive group;
- (ii)  $\rho$  is faithful;
- (iii) the images of all  $G(E)$ -conjugates of elements of  $\Psi$  generate  $G_E$ .

The *weights* of a Mumford–Tate triple  $(G, \rho, \Psi)$  are the integers which appear as weights of  $\rho \circ \nu$  (a representation of  $\mathbb{G}_m$ ) for some  $\nu \in \Psi$ .

A weak Mumford–Tate triple  $(G, \rho, \Psi)$  is called *pure* if  $\rho(G)$  contains the torus  $\mathbb{G}_m \cdot \text{id}$  of homotheties.

*The Mumford–Tate group.*— Let  $A$  be an abelian variety over  $\mathbb{C}$ , of dimension  $g$ . The singular cohomology group  $H^1(A(\mathbb{C}), \mathbb{Q})$  is a vector space of dimension  $2g$  over  $\mathbb{Q}$ . Hodge theory gives a decomposition of  $\mathbb{C}$ -vector spaces

$$H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^{1,0}(A) \oplus H^{0,1}(A)$$

with  $H^{1,0}(A)$  and  $H^{0,1}(A)$  being mapped onto each other by complex conjugation (so each has dimension  $g$ ).

We define a cocharacter  $\mu : \mathbb{G}_{m, \mathbb{C}} \rightarrow \text{GL}_{2g, \mathbb{C}}$  by:

$$\begin{aligned} \mu(z) \text{ acts as multiplication by } z \text{ on } H^{1,0}(A) \\ \text{and as the identity on } H^{0,1}(A). \end{aligned}$$

The *Mumford–Tate group* of  $A$  is defined to be the smallest algebraic subgroup  $M$  of  $\text{GL}_{2g}$  defined over  $\mathbb{Q}$  and such that  $M_{\mathbb{C}}$  contains the image of  $\mu$ .

The triple consisting of the Mumford–Tate group, its defining representation  $\rho : M \rightarrow \text{GL}_{2g}$ , and the set of  $\text{Aut}(\mathbb{C}/\mathbb{Q})$ -conjugates of the cocharacter  $\mu$  form a pure weak Mumford–Tate triple of weights  $\{0, 1\}$ . This is immediate from the definitions.

*The  $\ell$ -adic algebraic monodromy group.*— Now suppose that the abelian variety  $A$  is defined over a number field  $K$ . Its first  $\ell$ -adic cohomology group is a  $\mathbb{Q}_{\ell}$ -vector space of dimension  $2g$ , isomorphic to the dual of the  $\ell$ -adic Tate module:

$$H^1(A_{\bar{K}}, \mathbb{Q}_{\ell}) \cong (T_{\ell} A \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell})^{\vee}.$$

The Galois group  $\text{Gal}(\bar{K}/K)$  acts on the torsion points of  $A(\bar{K})$ , and this induces an action on  $H^1(A_{\bar{K}}, \mathbb{Q}_{\ell})$ , or in other words a continuous representation

$$\rho_{\ell} : \text{Gal}(\bar{K}/K) \rightarrow \text{GL}_{2g}(\mathbb{Q}_{\ell}).$$

The  *$\ell$ -adic algebraic monodromy group* of  $A$  is the smallest algebraic subgroup  $G_{\ell}$  of  $\text{GL}_{2g, \mathbb{Q}_{\ell}}$  whose  $\mathbb{Q}_{\ell}$ -points contain the image of  $\rho_{\ell}$ . By working with the  $\ell$ -adic monodromy group instead of the image of  $\rho_{\ell}$  directly, we gain the ability to use the structure theory of algebraic groups. On the other hand,

we do not lose very much because  $\text{Im } \rho_\ell$  is known [Bog81] to be an open (and hence finite-index) subgroup of  $G_\ell(\mathbb{Q}_\ell) \cap \text{GL}_{2g}(\mathbb{Z}_\ell)$ .

Pink [Pin98] has proved that the identity component  $G_\ell^\circ$  together with the representation  $\rho_\ell$  and a certain set  $\Psi$  of cocharacters form a pure weak Mumford–Tate triple of weights  $\{0, 1\}$ .

### 3. The commutative endomorphism case

Let  $(G, \rho, \Psi)$  be a pure weak Mumford–Tate triple of weights  $\{0, 1\}$ , and let  $T$  be a maximal torus of  $G$ . In this section we will give an upper bound for the number of distinct characters in  $\rho|_T$  as a function of  $\text{rk } G$ . We deduce Theorem 1.1, giving a lower bound for the rank of the Mumford–Tate group of an abelian variety with a commutative endomorphism ring.

If  $A$  has complex multiplication (in other words if  $G$  is a torus) then this bound was obtained by Ribet [Rib80]. Our method of proving the bound is inspired by that of Ribet, applied to a maximal torus of  $G$ , although it is convenient to arrange it differently.

**PROPOSITION 3.1.** — *Let  $(G, \rho, \Psi)$  be a pure weak Mumford–Tate triple of weights  $\{0, 1\}$ . The number of distinct characters in  $\rho|_T$  is at most  $2^{\text{rk } G - 1}$ .*

*Proof.* — Let  $Y = \text{Hom}(\mathbb{G}_{m,E}, T_E) \otimes_{\mathbb{Z}} \mathbb{Q}$  be the quasi-cocharacter space of  $T$ .

Let  $\Psi'$  be the set of all cocharacters of  $T_E$  which are  $G(E)$ -conjugate to an element of  $\Psi$ . Every cocharacter of  $G$  has a  $G(E)$ -conjugate whose image is contained in  $T_E$ , so  $\Psi'$  still satisfies condition (iii) in the definition of a weak Mumford–Tate triple. Replacing  $\Psi$  by  $\Psi'$  does not change the weights of our Mumford–Tate triple.

Furthermore  $\Psi'$  is closed under the action of the Weyl group of  $G_E$  on  $Y$ . So condition (iii) implies that  $\Psi'$  spans  $Y$  as a  $\mathbb{Q}$ -vector space.

Let  $\Theta$  be a basis of  $Y$  contained in  $\Psi'$ . The character space of  $T$  is dual to  $Y$ , so any character  $\omega$  is determined by its inner products  $\langle \omega, \mu \rangle$  for  $\mu \in \Theta$ .

Because our Mumford–Tate triple has weights  $\{0, 1\}$ , if  $\mu$  is a character in  $\rho|_T$  then these inner products can only have the values 0 or 1. So there are at most  $2^{|\Theta|}$  distinct characters in  $\rho|_T$ , and  $|\Theta| = \text{rk } G$ .

We can use the fact that our Mumford–Tate triple is pure to improve the exponent to  $\text{rk } G - 1$ . We know that  $\rho(G)$  contains the homotheties. Since  $\rho$  is faithful, there is a unique cocharacter  $\mu_0 : \mathbb{G}_m \rightarrow G$  such that  $\rho \circ \mu_0(z) = z \cdot \text{id}$ . We take  $\Theta'$  to be a subset of  $\Psi$  such that  $\Theta' \cap \{\mu_0\}$  is a basis of  $Y$ . Now  $\langle \omega, \mu_0 \rangle = 1$  for all characters  $\omega$  in  $\rho|_T$ , so  $\omega$  is determined by the values  $\langle \omega, \mu \rangle$  for  $\mu \in \Theta'$ . We may repeat the previous argument with  $\Theta$  replaced by  $\Theta'$ .  $\square$

**PROPOSITION (Theorem 1.1).** — *Let  $A$  be an abelian variety of dimension  $g$  such that  $\text{End } A$  is commutative. Let  $G$  be the Mumford–Tate group or the  $\ell$ -adic monodromy group of  $A$ . Then  $\text{rk } G \geq \log_2 g + 2$ .*

*Proof.* — We know that  $G$  forms part of a pure weak Mumford–Tate triple  $(G, \rho, \Psi)$  of weights  $\{0, 1\}$ .

Let  $F$  be the field of definition of  $\rho$ , i.e.  $\mathbb{Q}$  if  $G$  is the Mumford–Tate group and  $\mathbb{Q}_\ell$  if  $G$  is the  $\ell$ -adic monodromy group. Let  $\Sigma$  be the set of absolutely irreducible components of  $\rho \otimes_F \bar{F}$  and for each  $\sigma \in \Sigma$ , let  $m_\sigma$  be the multiplicity of  $\sigma$  in  $\rho \otimes_F \bar{F}$ .

Serre [Ser79] showed that each absolutely irreducible component  $\sigma$  is minuscule, i.e. the characters in  $\sigma|_T$  form a single orbit under the action of the Weyl group. Serre only treated strong Mumford–Tate triples, but his argument remains valid for weak Mumford–Tate triples (see also [Pin98] Section 4 and [Zar84]).

The characters of  $T$  in a minuscule representation have multiplicity 1, and non-isomorphic minuscule representations contain disjoint characters. So the multiplicity of any character in  $\rho|_T$  is equal to the multiplicity of the unique irreducible component  $\sigma$  which contains that character, and so

$$\dim \rho \leq (\max\{m_\sigma \mid \sigma \in \Sigma\}) \cdot (\text{the number of distinct characters in } \rho|_T).$$

We know that  $\text{End } \rho = \text{End } A \otimes_{\mathbb{Z}} F$ . In particular  $\text{End } \rho$  is commutative, and so all the multiplicities  $m_\sigma$  are 1. Using Proposition 3.1 we deduce that

$$2g = \dim \rho \leq 2^{\text{rk } G - 1}. \quad \square$$

#### 4. The noncommutative endomorphism case

Let  $A$  be an abelian variety,  $G$  its Mumford–Tate group or  $\ell$ -adic monodromy group,  $\rho$  the associated representation and  $F = \mathbb{Q}$  or  $\mathbb{Q}_\ell$  the field of definition of  $G$ . In this section we will bound the multiplicities of absolutely irreducible components of  $\rho \otimes_F \bar{F}$ . Specifically, if  $n = \text{rk } G$ , then each absolutely irreducible component has rank at most  $\alpha(n)\sqrt{n \log_e n}$  for a function  $\alpha(n)$  satisfying the conditions of Theorem 1.2. As in the proof of Theorem 1.1 in section 3, such a bound together with Proposition 3.1 implies Theorem 1.2.

To establish this bound, we introduce an invariant  $u(G)$  for a reductive group  $G$  such that for any  $F$ -irreducible representation of  $G$ , the multiplicity of its irreducible components over  $\bar{F}$  is at most  $u(G)$ . Then we use Landau’s function (the maximum LCM of a set of positive integers with given sum) to obtain a bound for  $u(G)$ .

##### 4.1. Multiplicity of irreducible representations and $u(G)$ . —

DEFINITION. — Let  $G$  be a reductive group defined over the field  $F$ . Let  $T$  be a maximal torus of  $G$  and  $\Lambda = \text{Hom}(T_{\bar{F}}, \mathbb{G}_m)$  the character group of  $T$ . Let  $\Lambda_0$  be the subgroup of  $\Lambda$  generated by the roots of  $G$  and characters which vanish on  $T \cap G^{\text{der}}$ .

We define  $u(G)$  to be the exponent of  $\Lambda/\Lambda_0$ . (Observe that  $\Lambda/\Lambda_0$  is a finite abelian group, dual to the centre of  $G^{\text{der}}(\bar{F})$ .)

LEMMA 4.1. — *Let  $G$  be a reductive group over a field  $F$  and  $\rho$  an  $F$ -irreducible representation of  $G$ . Let  $D$  be the endomorphism ring of  $\rho$  and  $E$  the centre of  $D$ . Then the order of  $[D]$  in  $\text{Br } E$  divides  $u(G)$ .*

*Proof.* — Fix a base  $\Delta$  for the root system of  $G$  with respect to  $T$ . When we refer to the action of  $\text{Gal}(\bar{F}/F)$  on  $\Lambda$  below, this is the natural action twisted by the Weyl group so that it preserves the set  $\Delta$  (this is the same action used in [Tit71]).

Let  $\sigma$  be an absolutely irreducible component of  $\rho \otimes_F \bar{F}$ , and  $\lambda \in \Lambda$  the highest weight of  $\sigma$ . Let  $\Gamma$  be the subgroup of  $\text{Gal}(\bar{F}/F)$  fixing  $\lambda$ . Then  $E$  is isomorphic to the subfield of  $\bar{F}$  fixed by  $\Gamma$ .

Tits ([Tit71] Corollary 3.5) defined a homomorphism

$$\alpha_{G,E} : \Lambda^\Gamma \rightarrow \text{Br } E$$

such that  $\alpha_{G,E}(\lambda) = [D]^{-1}$ . He showed that the kernel of  $\alpha_{G,E}$  contains  $\Lambda_0^\Gamma$ .

Hence the order of  $[D]$  in  $\text{Br } E$  divides the exponent of  $\Lambda^\Gamma/\Lambda_0^\Gamma$ . But the latter is a subgroup of  $\Lambda/\Lambda_0$ , so its exponent divides  $u(G)$ .  $\square$

COROLLARY 4.2. — *Let  $G$  be a reductive group defined over a number field or a local field  $F$ . Let  $\rho$  be an  $F$ -irreducible representation of  $G$ . Then the multiplicity of each absolutely irreducible component of  $\rho \otimes_F \bar{F}$  divides  $u(G)$ .*

*Proof.* — Let  $D = \text{End } \rho$  and let  $E$  be the centre of  $D$ . Then the multiplicity of any absolutely irreducible component of  $\rho \otimes_F \bar{F}$  is  $\sqrt{\dim_E D}$ .

Since  $F$  is a number field or a local field, it follows from class field theory that  $\sqrt{\dim_E D}$  is equal to the order of  $[D]$  in  $\text{Br } E$  (see e.g. [Pie82] Theorem 18.6).

Now apply Lemma 4.1.  $\square$

In the case of the Mumford–Tate group, if the abelian variety  $A$  is simple then  $\rho$  is  $\mathbb{Q}$ -irreducible. So Corollary 4.2 suffices to prove that the multiplicity of an absolutely irreducible component of  $\rho \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$  is bounded above by  $u(G)$ .

In the case of the  $\ell$ -adic monodromy group, we have to work harder, as a simple abelian variety may have reducible  $\ell$ -adic representations. However by Faltings’ Theorem the multiplicities of absolutely irreducible components are still equal to the index of the  $\mathbb{Q}$ -algebra  $\text{End } A$  (which is independent of  $\ell$ ). We will use results of Serre and Pink to show that  $u(G_\ell^\circ)$  is also independent of  $\ell$ , and then we can consider all  $\ell$  at once to show that the index of  $\text{End } A$  is bounded above by  $u(G_\ell^\circ)$ .

LEMMA 4.3. —  *$u(G_\ell^\circ)$  is independent of  $\ell$ .*

*Proof.* — Let  $\ell, \ell'$  be any two rational primes. Via  $\rho_\ell$ , we view  $G_\ell^\circ$  as a subgroup of  $\mathrm{GL}_{2g, \mathbb{Q}_\ell}$ .

For a finite place  $v$  of  $K$ , let  $T_v$  be the Frobenius torus of  $A$  in the sense of Serre [Ser81]. Serre showed that we can choose  $v$  such that  $T_{v, \mathbb{Q}_\ell}$  is  $\mathrm{GL}_{2g, \mathbb{Q}_\ell}$ -conjugate to a maximal torus of  $G_\ell^\circ$ , and such that the analogous property holds for  $\ell'$ .

Hence we get maximal tori  $T_{v, \ell}$  of  $G_\ell$  and  $T_{v, \ell'}$  of  $G_{\ell'}$  together with an isomorphism  $\Lambda(T_{v, \ell}) \cong \Lambda(T_v) \cong \Lambda(T_{v, \ell'})$ . Furthermore, under this isomorphism, the formal character of  $\rho_\ell$  corresponds to the formal character of  $\rho_{\ell'}$ .

As observed by Larsen-Pink [LP90], the formal character of a faithful irreducible representation of a reductive group determines the root lattice  $\Lambda_0$ . Hence  $\Lambda/\Lambda_0(G_\ell^\circ) \cong \Lambda/\Lambda_0(G_{\ell'}^\circ)$  so  $u(G_\ell^\circ) = u(G_{\ell'}^\circ)$ .  $\square$

**PROPOSITION 4.4.** — *Let  $A$  be a simple abelian variety defined over a number field, and  $G_\ell$  its  $\ell$ -adic monodromy group. The multiplicity of every absolutely irreducible component of  $\rho_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell$  divides  $u(G_\ell^\circ)$ .*

*Proof.* — Let  $D = \mathrm{End} A \otimes_{\mathbb{Z}} \mathbb{Q}$  be the endomorphism algebra of  $A$ , and let  $E$  be the centre of  $D$ . Let  $m^2 = \dim_E D$ .

By Faltings' Theorem,  $\mathrm{End} \rho_\ell = D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ . This is a product of algebras, each of dimension  $m^2$  over its centre. So every absolutely irreducible component of  $\rho_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell$  has multiplicity  $m$ , and it will suffice to show that  $m$  divides  $u(G_\ell^\circ)$ .

By Albert's classification of endomorphism rings of abelian varieties,  $E$  is either totally real or a CM field, and if  $E$  is totally real then  $m \leq 2$ . Hence we may assume that  $E$  is a CM field.

For each place  $\lambda$  of  $E$ ,  $D_\lambda = D \otimes_E E_\lambda$  is a matrix ring over a division algebra over  $E_\lambda$ . Let  $m_\lambda$  be the order of  $[D_\lambda]$  in  $\mathrm{Br} E_\lambda$ . We know that the map  $[D] \mapsto ([D_\lambda])$  is an injection

$$\mathrm{Br} E \rightarrow \bigoplus_{\lambda} \mathrm{Br} E_\lambda$$

so  $m$  is the lowest common multiple of the  $m_\lambda$ . So it suffices to show that  $m_\lambda$  divides  $u(G_\ell^\circ)$  for every place  $\lambda$ .

Since  $E$  is a CM field, all its archimedean places have trivial Brauer group, so we need only consider non-archimedean places. Let  $\lambda$  be a non-archimedean place of  $E$  and  $\ell'$  its residue characteristic. Then

$$\mathrm{End} \rho_{\ell'} = D \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell'} = D \otimes_E \left( \prod_{\lambda' | \ell'} E_{\lambda'} \right) = \prod_{\lambda' | \ell'} D_{\lambda'}.$$

Hence  $\rho_{\ell'}$  has a  $\mathbb{Q}_{\ell'}$ -irreducible subrepresentation with endomorphism algebra  $D_\lambda$ .

So by Lemma 4.1,  $m_\lambda$  divides  $u(G_{\ell'}^\circ)$ , and this is equal to  $u(G_\ell^\circ)$  by Lemma 4.3.  $\square$



#### 4.2. Bounds for $u(G)$ . —

DEFINITION. — Let  $g(n)$  be the maximum value of  $\text{LCM}(a_i)$  where  $a_i$  are positive integers satisfying  $\sum a_i = n$ . (This is Landau’s function.)

Let  $g_1(n)$  be the maximum value of  $\text{LCM}(a_i)$  where  $a_i$  are integers greater than 1 satisfying  $\sum (a_i - 1) = n$ .

For  $n \geq 2$ , let

$$\alpha(n) = \frac{\log_2 g_1(n)}{\sqrt{n \log n}}.$$

LEMMA 4.5. — For any reductive group  $G$ ,  $u(G) \leq g_1(\text{rk } G)$ .

*Proof.* — Let  $\Phi_i$  (for  $i \in I$ ) be the simple components of the root system of  $G$ .

The group  $\Lambda/\Lambda_0$  is a subgroup of the product of the fundamental groups of the  $\Phi_i$ . So  $u(G)$  divides the lowest common multiple of the exponents of these fundamental groups.

Let  $e_i$  be the exponent of the fundamental group of  $\Phi_i$ . Then  $e_i \leq \text{rk } \Phi_i + 1$  for all  $i$  (by the classification of simple root systems), and so  $\sum_i (e_i - 1) \leq \text{rk } G$ .

By the definition of  $g_1$ ,

$$u(G) \leq g_1\left(\sum_{i \in I} (e_i - 1)\right) \leq g_1(\text{rk } G)$$

and this is less than or equal to  $g_1(\text{rk } G)$  because  $g_1$  is nondecreasing.  $\square$

COROLLARY 4.6. —  $\alpha(n) \rightarrow \frac{1}{\log 2}$  as  $n \rightarrow \infty$  and  $\alpha(n) < 2$  for all  $n \geq 2$ .

*Proof.* — We use two results on the size of  $g(n)$ : Landau’s asymptotic result [Lan09]

$$\frac{\log_e g(n)}{\sqrt{n \log n}} \rightarrow 1 \text{ as } n \rightarrow \infty$$

and Massias’ bound [Mas84]

$$\log_e g(n) < 1.05314 \sqrt{n \log n} \text{ for all } n \geq 2.$$

We note that  $g(n) \leq g_1(n) \leq g(n + \lfloor \sqrt{2n} \rfloor)$  since any set of distinct positive integers satisfying  $\sum_i (a_i - 1) = n$  will satisfy  $\sum_i a_i \leq n + \lfloor \sqrt{2n} \rfloor$ .

Let

$$f(x) = \frac{(x + \sqrt{2x}) \log(x + \sqrt{2x})}{x \log x}.$$

Since  $f(x) \rightarrow 1$  as  $x \rightarrow \infty$ , we conclude that  $\alpha(n) \rightarrow \frac{1}{\log 2}$ .

Likewise by Massias’ bound

$$\alpha(n) \leq \frac{\log_e g(n + \lfloor \sqrt{2n} \rfloor)}{\log 2 \sqrt{n \log n}} < \frac{1.05314 \sqrt{f(n)}}{\log 2} \leq \frac{1.05314 \sqrt{f(9)}}{\log 2} < 2$$

for  $n \geq 9$  since  $f(x)$  is decreasing for  $x > 1$ .

Manual calculation shows that  $\alpha(n) < 2$  for  $2 \leq n \leq 8$ .  $\square$

## 5. Some examples

In this section, we will give three examples of families of abelian varieties with commutative endomorphism ring for which Theorem 1.1 is sharp. We also give one family of simple abelian varieties with noncommutative endomorphism ring for which the Mumford–Tate group has rank  $n$  and the dimension  $g$  satisfies  $\log_2 g = n + \frac{1}{2} \log_2 n + O(1)$ . This shows that the bound in Theorem 1.2 cannot be improved to  $n \geq \log_2 g + O(1)$ .

### 5.1. Examples with commutative endomorphism ring. —

*Example 1: Complex multiplication.*— Let  $F$  be a totally real field such that  $[F : \mathbb{Q}] = n - 1$ . By [Shi70] Theorem 1.10, there is an imaginary quadratic extension  $K$  of  $F$  such that for every CM type  $(K, \Phi)$ , the reflex type  $(K', \Phi')$  satisfies  $[K' : \mathbb{Q}] = 2^{n-1}$ . Such a CM type is primitive.

Let  $A$  be a complex abelian variety corresponding to the CM type  $(K', \Phi')$ . Then the Mumford–Tate group  $M$  is a torus, isomorphic to the image of the homomorphism  $\text{Res}_{K/\mathbb{Q}} \mathbb{G}_m \rightarrow \text{Res}_{K'/\mathbb{Q}} \mathbb{G}_m$  induced by the reflex norm  $K^\times \rightarrow K'^\times$ .

This image has rank at most  $[K : \mathbb{Q}] + 1 = n + 1$ . But  $\dim A = 2^{n-1}$  so by Theorem 1.1,  $\text{rk } M \geq n + 1$ . So in fact  $\text{rk } M = n + 1 = \log_2 \dim A + 2$ .

The endomorphism ring of  $A$  is the field  $K'$ .

*Example 2: Spin group.*— This example generalises the Kuga–Satake construction of an abelian variety attached to a polarised  $K3$  surface [KS67].

Let  $n$  be a positive integer congruent to 1 or 2 mod 4. Let  $W$  be a  $\mathbb{Q}$ -vector space of dimension  $2n + 1$ , and let  $Q$  be the quadratic form

$$Q(x) = x_1^2 + x_2^2 - x_3^2 - \cdots - x_{2n+1}^2$$

of signature  $(2, 2n - 1)$ . The even Clifford algebra  $C^+(W, Q)$  is isomorphic to  $M_{2^n}(\mathbb{Q})$ , and so it has a unique faithful irreducible  $\mathbb{Q}$ -representation of dimension  $2^n$ , called the spin representation.

Let  $M$  be the Clifford group

$$\text{GSpin}(W, Q) = \{x \in C^+(W, Q) \mid xWx^{-1} \subseteq W\}.$$

This is a reductive group of rank  $n + 1$ , with root system  $B_n$  and centre  $\mathbb{G}_m$ . Let  $\rho : M \rightarrow \text{GL}(V)$  be the spin representation of  $M$ . This is an absolutely irreducible representation of dimension  $2^n$ .

Let  $\{e_1, e_2\}$  be an orthonormal basis for the positive definite subspace of  $W$ . The homomorphism  $\varphi : \mathbb{C}^\times \rightarrow M(\mathbb{R})$  given by

$$\varphi(a + ib) = a + be_1e_2$$

defines a Hodge structure on  $V$  of type  $\{(0, -1), (-1, 0)\}$ . The conditions on  $n \bmod 4$  and on the signature of  $W$  ensure that this Hodge structure is polarisable.

Because  $M^{\text{der}}$  is almost simple, replacing  $\varphi$  by a generic  $M(\mathbb{R})$ -conjugate gives a Hodge structure whose Mumford–Tate group is  $M$ . Let  $A$  be a complex abelian variety corresponding to such a Hodge structure. It has dimension  $2^{n-1}$  and endomorphism algebra  $\mathbb{Q}$ , and its Mumford–Tate group has rank  $n + 1$ .

*Example 3: Product of copies of  $\text{SL}_2$ .*— This example generalises the example of Mumford [Mum69] of a family of abelian varieties of dimension 4 with Mumford–Tate group  $M$  such that  $M_{\mathbb{C}}$  is isogenous to  $\mathbb{G}_m \times (\text{SL}_2)^3$ .

Let  $n$  be an odd positive integer, and  $F$  a totally real number field of degree  $n$ . Let  $D$  be a quaternion algebra over  $F$  such that:

- (i)  $\text{Cor}_{F/\mathbb{Q}} D$  is split over  $\mathbb{Q}$ , i.e. is isomorphic to  $M_{2^n}(\mathbb{Q})$ .
- (ii)  $D$  is split at exactly one real place of  $F$ .

Let  $M$  be the  $\mathbb{Q}$ -algebraic group  $M(A) = \{x \in (D \otimes A)^{\times} \mid x\bar{x} \in A^{\times}\}$  (where  $\bar{x}$  is the standard involution of  $D$ ). By condition (ii),  $M_{\mathbb{R}}$  is isomorphic to

$$(\mathbb{G}_{m,\mathbb{R}} \times \text{SL}_{2,\mathbb{R}} \times \text{SU}_2^{n-1}) / \{(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \mid \varepsilon_i \in \{\pm 1\}, \varepsilon_0 \varepsilon_1 \cdots \varepsilon_n = 1\}.$$

By condition (i),  $M$  has a faithful irreducible  $\mathbb{Q}$ -representation  $\rho$  of dimension  $2^n$ . Then  $\rho \otimes_{\mathbb{Q}} \mathbb{C}$  is isomorphic to the tensor product of the standard 1-dimensional representation of  $\mathbb{G}_{m,\mathbb{C}}$  with the standard 2-dimensional representation of each factor  $\text{SL}_{2,\mathbb{C}}$ .

Let  $\varphi : \mathbb{C}^{\times} \rightarrow M(\mathbb{R})$  be the homomorphism

$$\varphi(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ in } \text{GL}_2 \cong (\mathbb{G}_m \times \text{SL}_2) / \{\pm 1\}$$

and trivial in the  $\text{SU}_2$  factors.

Then  $\rho \circ \varphi$  defines a Hodge structure of type  $\{(0, -1), (-1, 0)\}$ . By condition (ii), this Hodge structure is polarisable.

Again  $M^{\text{der}}$  is almost  $\mathbb{Q}$ -simple, so replacing  $\varphi$  by a generic element of its  $M(\mathbb{R})$ -conjugacy class gives a Hodge structure with Mumford–Tate group equal to  $M$ . An abelian variety corresponding to such a Hodge structure will have dimension  $2^{n-1}$ , endomorphism algebra  $\mathbb{Q}$  and Mumford–Tate group of rank  $n + 1$ .

**5.2. An example with large multiplicity.** — Let  $n$  be an odd integer and  $r = (n - 1)/2$ . We will construct a simple abelian variety of dimension  $g(n) = n \binom{n}{r}$  whose Mumford–Tate group is a  $\mathbb{Q}$ -form of  $\text{GL}_n$ . The Mumford–Tate representation is isomorphic over  $\mathbb{C}$  to the sum of  $2n$  copies of the  $r$ -th exterior power of the standard representation. By Stirling’s formula  $\log_2 g(n) = n + \frac{1}{2} \log_2 n + O(1)$ .

Let  $K$  be an imaginary quadratic field, and  $D$  a central division algebra over  $K$  of dimension  $n^2$  with an involution  $*$  of the second kind. The  $\mathbb{Q}$ -algebraic

groups

$$\begin{aligned} H(A) &= \{d \in (D \otimes_{\mathbb{Q}} A)^{\times} \mid dd^* = 1\}, \\ G(A) &= \{d \in (D \otimes_{\mathbb{Q}} A)^{\times} \mid dd^* \in A^{\times}\} \end{aligned}$$

are  $\mathbb{Q}$ -forms of  $\mathrm{SL}_n$  and  $\mathrm{GL}_n$ . By choosing  $*$  appropriately, we may suppose that  $H_{\mathbb{R}}$  is the unitary group of a Hermitian form of signature  $(1, n-1)$ .

We can view  $D$  as a  $K$ -irreducible representation of  $H_K$ . Over  $\mathbb{C}$ ,  $D_{\mathbb{C}}$  is isomorphic to the sum of  $n$  copies of the standard representation of  $\mathrm{SL}_n$ , so its highest weight is  $\varpi_1$ . The endomorphism ring of this representation is  $D^{op}$ , so

$$\alpha_{H,K}(\varpi_1) = [D]$$

for Tits' homomorphism  $\alpha_{H,K} : \Lambda^{\Gamma} \rightarrow \mathrm{Br} K$ .

Let  $r = (n-1)/2$  and let  $\tilde{D}$  be the central division algebra over  $K$  such that  $[\tilde{D}] = [D]^r$  in  $\mathrm{Br} K$ . Now  $[D]$  has order  $n$  in  $\mathrm{Br} K$ . Since  $r$  and  $n$  are coprime,  $[\tilde{D}]$  also has order  $n$  and  $\tilde{D} \otimes_K \mathbb{C} \cong M_n(\mathbb{C})$ .

Let  $\tilde{\rho}$  be the  $K$ -irreducible representation of  $H_K$  with highest weight  $\varpi_r$ . We know that  $\varpi_r \equiv r\varpi_1$  modulo the roots of  $H_K$ , so  $\alpha_{H,K}(\varpi_r) = [D]^r = [\tilde{D}]$ . Hence  $\tilde{\rho}$  has endomorphism ring  $\tilde{D}^{op}$ , so  $\tilde{\rho}_{\mathbb{C}}$  is the sum of  $n$  copies of an irreducible representation of  $\mathrm{SL}_n$ . This irreducible representation is the  $r$ -th exterior power of the standard representation, so  $\dim_K \tilde{\rho} = n \binom{n}{r}$ .

If  $\lambda I$  is a scalar matrix in  $H(\mathbb{C})$ , then  $\tilde{\rho}_{\mathbb{C}}(\lambda I)$  is multiplication by  $\lambda^r$ . So we can extend  $\tilde{\rho}$  to a representation of  $G_K$  by letting each scalar matrix  $\lambda I$  act as multiplication by  $\lambda^r$ .

Let  $\rho = \mathrm{Res}_{K/\mathbb{Q}} \tilde{\rho}$ . This is a  $\mathbb{Q}$ -irreducible representation of  $G$  of dimension  $2n \binom{n}{r}$ . We have  $\ker \rho = \mu_r$  so  $\rho$  factorises through  $M = G/\mu_r$ , and the resulting representation of  $M$  is faithful.

In order to specify the Hodge structure, we will first define  $\varphi' : \mathbb{C}^{\times} \rightarrow G(\mathbb{R})$  as follows: recall that  $H_{\mathbb{R}}$  is the unitary group of a Hermitian form  $\Psi$  of signature  $(1, n-1)$ . Then let  $\phi'(z)$  act as  $z^r/\bar{z}^{r-1}$  on the 1-dimensional space where  $h$  is positive definite and as  $\bar{z}$  on the  $(n-1)$ -dimensional space where  $h$  is negative definite.

Then  $\rho \circ \varphi'$  has weights  $z^r$  and  $\bar{z}^r$ . Because  $\rho$  is faithful as a representation of  $M$ , it follows that there is a homomorphism  $\varphi : \mathbb{C}^{\times} \rightarrow M(\mathbb{R})$  whose  $r$ -th power is  $\varphi'$ . Then  $(M, \rho, \varphi)$  defines a  $\mathbb{Q}$ -Hodge structure of type  $\{(-1, 0), (0, -1)\}$ . The Hermitian form  $\Psi$  induces a polarisation of this Hodge structure.

Once again,  $M^{\mathrm{der}}$  is almost simple, so replacing  $\varphi$  by a generic  $M(\mathbb{R})$ -conjugate gives a Hodge structure with Mumford–Tate group  $M$ . A corresponding abelian variety will have endomorphism algebra  $\tilde{D}^{op}$  and dimension  $g = n \binom{n}{r}$ .

## BIBLIOGRAPHY

- [Bog81] F. A. BOGOMOLOV – “Points of finite order on an abelian variety”, *Izvestia Mathematics* **17** (1981), no. 1, p. 55–72.
- [Dod87] B. DODSON – “On the Mumford-Tate group of an abelian variety with complex multiplication”, *Journal of Algebra* **111** (1987), no. 1, p. 49–73.
- [KS67] M. KUGA & I. SATAKE – “Abelian varieties attached to polarized  $K_3$ -surfaces”, *Mathematische Annalen* **169** (1967), p. 239–242.
- [Lan09] E. LANDAU – *Handbuch der lehre von der verteilung der primzahlen*, vol. I, 1909.
- [LP90] M. J. LARSEN & R. PINK – “Determining representations from invariant dimensions”, *Inventiones Mathematicae* **102** (1990), p. 377–398.
- [Mas84] J. MASSIAS – “Majoration explicite de l’ordre maximum d’un élément du groupe symétrique”, *Annales Faculté des Sciences Toulouse Math.* **6** (1984), p. 269–291.
- [Mum69] D. MUMFORD – “A note on Shimura’s paper ‘Discontinuous groups and abelian varieties’”, *Mathematische Annalen* **181** (1969), p. 345–351.
- [Pie82] R. S. PIERCE – *Associative algebras*, Springer-Verlag, 1982.
- [Pin98] R. PINK – “ $\ell$ -adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture”, *Journal für die reine und angewandte Mathematik* **495** (1998), p. 187–237.
- [Rib80] K. A. RIBET – “Division fields of abelian varieties with complex multiplication”, *Mémoires de la S. M. F.* (1980), p. 75–94.
- [Ser79] J.-P. SERRE – “Groupes algébriques associés aux modules de Hodge-Tate”, *Astérisque* **65** (1979), p. 155–187.
- [Ser81] ———, “Lettre à Ken Ribet de 1/1/1981”, 1981, p. 6–17.
- [Ser85] ———, “Résumés des cours au Collège de France”, 1984–85.
- [Shi70] G. SHIMURA – “On canonical models of arithmetic quotients of bounded symmetric domains”, *Annals of Mathematics* **91** (1970), p. 144–222.
- [Tit71] J. TITS – “Représentations linéaires irréductibles d’un groupe réductif sur un corps quelconques”, *Journal für die reine und angewandte Mathematik* **247** (1971), p. 196–220.
- [Win86] J.-P. WINTENBERGER – “Groupes algébriques associés à certaines représentations  $p$ -adiques”, *American Journal of Mathematics* **108** (1986), no. 6, p. 1425–1466.
- [Zar84] Y. ZARHIN – “Weights of simple lie algebras in the cohomology of algebraic varieties”, *Izvestiya: Mathematics* **24** (1984), no. 2, p. 245–281.